Analysis of the Klein-Gordon-Folk Operator

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Consider the initial-valued Klein-Gordon-Folk Equation,

$$\frac{\partial^2 u}{\partial t^2} - \Delta_x u + m^2 u = 0, \quad t > 0, \quad x \in \mathbb{R}^3$$
$$u|_{t=0} = u_0(\vec{x}), \quad \frac{\partial u}{\partial t}|_{t=0} = u_1(\vec{x})$$

A solution to this equation is a wave function that describes a free relativistic scalar particle of mass m. We wish to find a distributional solution to our equation, determine under what conditions it is a classical solution, and analyze its well-posedness.

1 Formulation of the generalized Cauchy problem.

Let $v(\vec{x},t) = \theta(t)u(\vec{x},t) \in \mathcal{D}'_+$. Then

$$\frac{\partial v}{\partial t} = \delta(t) \cdot u_0(\vec{x}) + \theta(t)\left\{\frac{\partial u}{\partial t}\right\}$$
$$\frac{\partial^2 v}{\partial t^2} = \delta'(t) \cdot u_0(\vec{x}) + \delta(t) \cdot u_1(\vec{x}) + \theta(t)\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$
$$\Delta_x v = \theta(t)\left\{\Delta_x u\right\}$$
$$\therefore \frac{\partial^2 v}{\partial t^2} - \Delta_x v + m^2 v = \delta(t) \cdot u_1(\vec{x}) + \delta'(t) \cdot u_0(\vec{x})$$

Now that we have formalized the generalized Cauchy problem, we need to solve it. We will do so by calculating it's corresponding causal Green's function. Taking the convolution of our Green's function with our inhomogeneity will result in a distributional solution to our equation.

2 Calculation of Green's function

A retarded (or causal) Green's function for the Klein-Gordon-Folk operator satisfies the following equation,

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_x + m^2\right) G(\vec{x},t) = \delta(t) \cdot \delta(\vec{x}), \quad G(\vec{x},t) = 0, \quad t < 0$$

If our Green's function exists as a temperate distribution, then for any $\epsilon > 0$,

$$G_\epsilon(x,t) = e^{-\epsilon t} G(x,t) \in \mathcal{S}' \quad \text{and} \quad G_\epsilon \to G \quad \text{in } \mathcal{S}'$$

G_ϵ is clearly a temperate distribution since it decays exponentially as $t \to \infty$ and $G_\epsilon = 0$ if $t < 0$, due to the causality of G.
Furthermore, G_ϵ satisfies the following equation,

$$\left(\left(\frac{\partial}{\partial t} + \epsilon\right)^2 - \Delta_x + m^2\right) G_\epsilon(\vec{x}, t) = \delta(t) \cdot \delta(\vec{x}), \quad G_\epsilon(\vec{x}, t) = 0, \quad t < 0$$

We can validate this by plugging in.

$$\begin{aligned}
\left(\left(\frac{\partial}{\partial t} + \epsilon\right)^2 - \Delta_x + m^2\right) G_\epsilon(\vec{x}, t) &= \left(\frac{\partial^2}{\partial t^2} + 2\epsilon\frac{\partial}{\partial t} + \epsilon^2 - \Delta_x + m^2\right) G_\epsilon(\vec{x}, t) \\
&= \left(\frac{\partial^2}{\partial t^2} + 2\epsilon\frac{\partial}{\partial t} + \epsilon^2 - \Delta_x + m^2\right) e^{-\epsilon t} G(\vec{x}, t) \\
&= e^{-\epsilon t}\left(\frac{\partial^2}{\partial t^2} - \Delta_x + m^2\right) G(\vec{x}, t) \\
&= e^{-\epsilon t} \cdot \delta(t) \cdot \delta(\vec{x}) \\
&= \delta(t) \cdot \delta(\vec{x})
\end{aligned}$$

Additionally, $\mathcal{F}[G_\epsilon(\vec{x}, t)] \to \mathcal{F}[G(\vec{x}, t)]$ in $\mathcal{S}'$ as $\epsilon \to 0^+$ by continuity of the Fourier transform and

$$\begin{aligned}
&\left((-iw)^2 - 2iw\epsilon + \epsilon^2 - (-i)^2 < \vec{k}, \vec{k} > + m^2\right) \mathcal{F}[G_\epsilon(\vec{x}, t)](\vec{k}, w) = 1 \\
\implies &\mathcal{F}[G_\epsilon(\vec{x}, t)](\vec{k}, w) = \frac{1}{|\vec{k}|^2 + m^2 - (w + i\epsilon)^2}
\end{aligned}$$

since $\mathcal{F}_t[\delta(t)] = 1$ and $\mathcal{F}_x[\delta(x)] = 1$.
Observe, for a Schwartz function $\varphi \in \mathcal{S}$,

$$(G_\epsilon, \varphi) = (\mathcal{F}^{-1}[\mathcal{F}[G_\epsilon]], \varphi) = (\mathcal{F}[G_\epsilon], \mathcal{F}^{-1}[\varphi])$$

The action of a temperate distribution on a Schwartz function converges absolutely and hence we can regularize the integral in any way we'd like. So,

$$(G_\epsilon(\vec{x}, t), \varphi) = -\frac{1}{(2\pi)^4} \lim_{R \to \infty} \int \int \int \int_{-R}^{R} \frac{1}{(w + i\epsilon)^2 - (|\vec{k}|^2 + m^2)} e^{-iwt} e^{-i\vec{k}\cdot\vec{x}} \varphi \, dw \, d^3k \, d^3x \, dt$$

Let's focus on the inner integral for now. Observe that we have singularities for $w = \pm\sqrt{|\vec{k}|^2 + m^2} - i\epsilon$. Observe that both poles are shifted down. This aligns with our goal of obtaining a retarded Green's function and is the reason we choose our specific regularization for G.

Observe what happens if we close our contour in the positive $\text{Im}(w)$ half-space as shown in Figure 2. Doing so completely avoids our poles, giving us residue values of 0. Further notice that by choosing this path, we must set $t < 0$. Otherwise, the integral over our arc does not converge if we bring the limit in. So for $t < 0$, $G(\vec{x}, t) = 0$, which can be applied by adding a step function $\theta(t)$ to our final result.

Now, what happens if we include our poles and close our contour in the negative $\text{Im}(w)$ half-space as shown in Figure 3. First, observe that our contour is oriented clockwise, so we have an added negative sign.

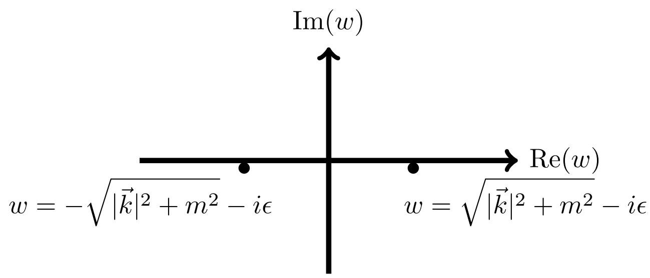

Figure 1: w-Plane with poles at $w = \pm\sqrt{|\vec{k}|^2 + m^2} - i\epsilon$.

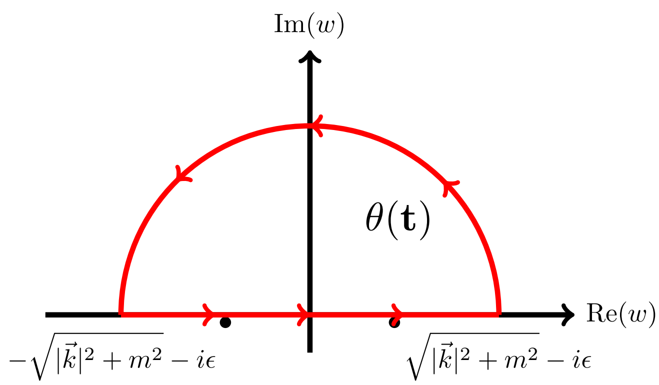

Figure 2: Contour closed over positive Im(w)

Using Cauchy's residue theorem, we obtain the following

$$\begin{aligned}
(G_\epsilon(\vec{x},t),\varphi) &= -\frac{1}{(2\pi)^4}\lim_{R\to\infty}\int\int\int\int_{-R}^{R}\frac{1}{(w+i\epsilon)^2-(|\vec{k}|^2+m^2)}e^{-iwt}e^{-i\vec{k}\cdot\vec{x}}\varphi\,dw\,d^3k\,d^3x\,dt \\
&= \frac{2\pi i\theta(t)}{(2\pi)^4}\lim_{a\to\infty}\int\int\int_{|k|<a}\left[\frac{e^{-i\sqrt{|\vec{k}|^2+m^2}t}}{2\sqrt{|\vec{k}|^2+m^2}}-\frac{e^{i\sqrt{|\vec{k}|^2+m^2}t}}{2\sqrt{|\vec{k}|^2+m^2}}\right]e^{-i\vec{k}\cdot\vec{x}}\varphi\,d^3k\,d^3x\,dt \\
&= \frac{\theta(t)}{i(2\pi)^3}\lim_{a\to\infty}\int\int\int_{|k|<a}\left[\frac{e^{i\sqrt{|\vec{k}|^2+m^2}t}}{2\sqrt{|\vec{k}|^2+m^2}}-\frac{e^{-i\sqrt{|\vec{k}|^2+m^2}t}}{2\sqrt{|\vec{k}|^2+m^2}}\right]e^{-i\vec{k}\cdot\vec{x}}\varphi\,d^3k\,d^3x\,dt
\end{aligned}$$

Note that we are allowed to bring the limit inside by Lebesgue Dominated Convergence Theorem since φ is a Schwartz function.

For the sake of clarity, we define the following two integrals as such:

$$I_1 = \int_{|k|<a}\frac{e^{i\sqrt{|\vec{k}|^2+m^2}t}}{2i\sqrt{|\vec{k}|^2+m^2}}e^{-i\vec{k}\cdot\vec{x}}\,d^3k$$

$$I_2 = \int_{|k|<a}\frac{e^{-i\sqrt{|\vec{k}|^2+m^2}t}}{2i\sqrt{|\vec{k}|^2+m^2}}e^{-i\vec{k}\cdot\vec{x}}\,d^3k$$

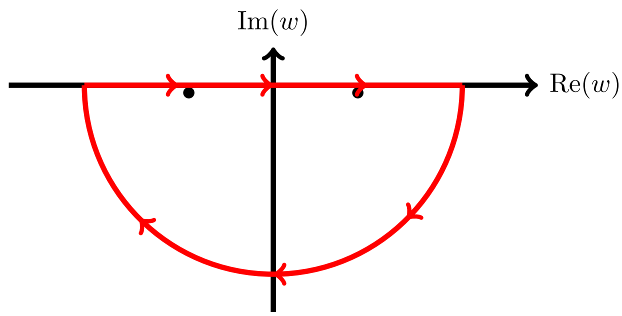

Figure 3: Contour closed over negative Im(w)

Since we are integrating over the entire space, making the substitution $k = -k$ yields

$$I_1 = \int_{|k|<a} \frac{e^{i\sqrt{|\vec{k}|^2+m^2}t}}{2i\sqrt{|\vec{k}|^2+m^2}} e^{i\vec{k}\cdot\vec{x}}\, d^3k$$

$$I_2 = \int_{|k|<a} \frac{e^{-i\sqrt{|\vec{k}|^2+m^2}t}}{2i\sqrt{|\vec{k}|^2+m^2}} e^{i\vec{k}\cdot\vec{x}}\, d^3k$$

First we will solve for I_1 and the solution for I_2 will follow since $I_2 = -I_1^\star$. Converting to spherical coordinates,

$$\begin{aligned} I_1 &= \int_{|k|<a} \frac{e^{i\sqrt{|\vec{k}|^2+m^2}t}}{2i\sqrt{|\vec{k}|^2+m^2}} e^{i\vec{k}\cdot\vec{x}}\, d^3k \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{e^{i\sqrt{|\vec{k}|^2+m^2}t}}{2i\sqrt{|\vec{k}|^2+m^2}} e^{i|\vec{k}||\vec{x}|\cos(\theta)}|\vec{k}|^2\sin(\theta)\, d|\vec{k}|\, d\theta\, d\phi \\ &= \frac{\pi}{|\vec{x}|}\int_0^{a} \frac{e^{i\sqrt{|\vec{k}|^2+m^2}t}}{\sqrt{|\vec{k}|^2+m^2}}|\vec{k}| \left(e^{-i|\vec{k}||\vec{x}|} - e^{i|\vec{k}||\vec{x}|}\right) d|\vec{k}| \\ &= \frac{i\pi}{|\vec{x}|}\frac{\partial}{\partial|\vec{x}|}\int_0^{a} \frac{e^{i\sqrt{|\vec{k}|^2+m^2}t}}{\sqrt{|\vec{k}|^2+m^2}} \left(e^{-i|\vec{k}||\vec{x}|} + e^{i|\vec{k}||\vec{x}|}\right) d|\vec{k}| \end{aligned}$$

We can interchange the order of integration to integrate over our angles first by Fubini's Theorem since everything converges absolutely (we are integrating over a bounded region). We can also take the partial derivative out of the integral by Fubini's Theorem since the integrand is bounded by $3/m$ and we are integrating over a bounded region. Even if $m = 0$, this is a removable singularity.
Making the following substitution, $|\vec{k}| = m\sinh(\xi)$,

$$I_1 = \frac{i\pi}{2|\vec{x}|}\frac{\partial}{\partial|\vec{x}|}\left[\int_0^{b(a)} e^{im[t\cosh(\xi)-|\vec{x}|\sinh(\xi)]}\, d\xi + \int_0^{b(a)} e^{im[t\cosh(\xi)+|\vec{x}|\sinh(\xi)]}\, d\xi\right]$$

where $b(a) = \sinh^{-1}(a/m)$.
There are four cases to be considered here:

1. $t > 0, \quad t > |\vec{x}|$

2. $t > 0, \quad t < |\vec{x}|$

3. $t < 0, \quad |t| > |\vec{x}|$

4. $t < 0, \quad |t| < |\vec{x}|$

Note that since our Green's function $G(\vec{x}, t) = 0, t < 0$, there is no point in evaluating our integral for case 3 or 4. To solve the integral in case 1 or 2, make the following substitutions:

1. $t = \sqrt{t^2 - |\vec{x}|^2} \cosh(z), \quad |\vec{x}| = \sqrt{t^2 - |\vec{x}|^2} \sinh(z)$

2. $t = \sqrt{|\vec{x}|^2 - t^2} \sinh(z), \quad |\vec{x}| = \sqrt{|\vec{x}|^2 - t^2} \cosh(z)$

These substitutions are valid because $\cosh^2(x) - \sinh^2(x) = 1$.
With these substitutions, we get the following (identities used can be found in appendix) for each case:
Case 1:

$$
\begin{aligned}
A_1 &= \frac{1}{2} \int_0^{b(a)} e^{im\sqrt{t^2 - |\vec{x}|^2}(\cosh(z)\cosh(\xi) - \sinh(z)\sinh(\xi))} \, d\xi + \frac{1}{2} \int_0^{b(a)} e^{im\sqrt{t^2 - |\vec{x}|^2}(\cosh(z)\cosh(\xi) + \sinh(z)\sinh(\xi))} \, d\xi \\
&= \frac{1}{2} \int_0^{b(a)} e^{im\sqrt{t^2 - |\vec{x}|^2}\cosh(\xi - z)} \, d\xi + \frac{1}{2} \int_0^{b(a)} e^{im\sqrt{t^2 - |\vec{x}|^2}\cosh(\xi + z)} \, d\xi \\
&= \int_0^{b(a)} e^{im\sqrt{t^2 - |\vec{x}|^2}\cosh(\xi)} \, d\xi \\
&= \frac{1}{2} \int_{-b(a)}^{b(a)} e^{im\sqrt{t^2 - |\vec{x}|^2}\cosh(\xi)} \, d\xi
\end{aligned}
$$

The last equality holds since $\cosh(x)$ is even.
Recall our limit for a, can we move it inside? Well, φ is a Schwartz function so we can move it inside subject to the innermost integral converging if we bring it inside. If doing so does cause the final integral to converge, then we would also be justified in moving the limit past the partial derivative. So would it converge? The answer is yes.

$$
\begin{aligned}
\lim_{a\to\infty} A_1 &= \frac{1}{2} \lim_{a\to\infty} \int_{-b(a)}^{b(a)} e^{im\sqrt{t^2 - |\vec{x}|^2}\cosh(\xi)} \, d\xi \\
&= \frac{1}{2} \lim_{b\to\infty} \int_{-b}^{b} e^{im\sqrt{t^2 - |\vec{x}|^2}\cosh(\xi)} \, d\xi \\
&= 2i\pi H_0^{(1)}(m\sqrt{t^2 - |\vec{x}|^2}) \\
&= 2i\pi J_0(m\sqrt{t^2 - |\vec{x}|^2}) - \pi N_0(m\sqrt{t^2 - |\vec{x}|^2})
\end{aligned}
$$

So we take our limit inside and obtain the expression in the last equality.
Case 2:

$$
\begin{aligned}
A_1 &= \frac{1}{2} \int_0^{b(a)} e^{im\sqrt{|\vec{x}|^2 - t^2}(\sinh(z)\cosh(\xi) - \cosh(z)\sinh(\xi))} \, d\xi + \frac{1}{2} \int_0^{b(a)} e^{im\sqrt{|\vec{x}|^2 - t^2}(\sinh(z)\cosh(\xi) + \cosh(z)\sinh(\xi))} \, d\xi \\
&= \frac{1}{2} \int_0^{b(a)} e^{im\sqrt{|\vec{x}|^2 - t^2}\sinh(\xi - z)} \, d\xi + \frac{1}{2} \int_0^{b(a)} e^{im\sqrt{|\vec{x}|^2 - t^2}\sinh(\xi + z)} \, d\xi \\
&= \int_0^{b(a)} e^{im\sqrt{|\vec{x}|^2 - t^2}\sinh(\xi)} \, d\xi \\
&= \frac{1}{2} \int_{-b(a)}^{b(a)} e^{im\sqrt{|\vec{x}|^2 - t^2}\sinh(\xi)} \, d\xi
\end{aligned}
$$

As in case 1, we can take the limit inside and obtain

$$\begin{aligned}
\lim_{a\to\infty} A_1 &= \frac{1}{2}\lim_{a\to\infty}\int_{-b(a)}^{b(a)} e^{im\sqrt{|\vec{x}|^2-t^2}\sinh(\xi)}\,d\xi \\
&= \frac{1}{2}\lim_{b\to\infty}\int_{-b}^{b} e^{im\sqrt{|\vec{x}|^2-t^2}\sinh(\xi)}\,d\xi \\
&= 2K_0(m\sqrt{|\vec{x}|^2-t^2})
\end{aligned}$$

From here, we can conclude that

$$A_1 = \begin{cases} 2i\pi J_0(m\sqrt{t^2-|\vec{x}|^2}) - \pi N_0(m\sqrt{t^2-|\vec{x}|^2}) & t > |\vec{x}| > 0 \\ 2K_0(m\sqrt{|\vec{x}|^2-t^2}) & |\vec{x}| > t > 0 \end{cases}$$

Taking derivative, multiplying by $\frac{i\pi}{2|\vec{x}|}$, taking conjugate and adding, we find that

$$I_1 - I_2 = \frac{2\pi^2}{|\vec{x}|}\delta(t-|\vec{x}|) - 2\pi^2 m\theta(t-|\vec{x}|)\frac{J_1(m\sqrt{t^2-|\vec{x}|^2})}{\sqrt{t^2-|\vec{x}|^2}}$$

where $J_0(z)$ is the Bessel function of order zero, $N_0(z)$ is the Neumann function of order zero, and $K_0(z)$ is the Hankel function of an imaginary argument of order zero.
And hence (recall the step function from before),

$$G(\vec{x},t) = \frac{\delta(t-|\vec{x}|)}{4\pi|\vec{x}|} - \theta(t-|\vec{x}|)\frac{mJ_1(m\sqrt{t^2-|\vec{x}|^2})}{4\pi\sqrt{t^2-|\vec{x}|^2}}, \quad G(\vec{x},t)=0, \quad t<0$$

3 Verifying initial conditions

Now that we have obtained our Green's function, we need to make sure that when we take the convolution of it with our inhomogeneity, we satisfy our initial conditions. To do so, we must see what it and its derivatives approach as t approaches 0^+. If we do so and they match, we know that we have obtained a valid distributional solution.

So, let $\varphi(\vec{x}) \in \mathcal{D}'(\mathbb{R}^3)$ be test function and let $g(t) = (G(\vec{x},t), \varphi(\vec{x}))$. Let us first compute $g(t)$ explicitly.

$$\begin{aligned}
g(t) &= (G(\vec{x},t),\varphi(\vec{x})) \\
&= \frac{1}{4\pi}\int_{|\vec{x}|=t}\frac{\varphi(\vec{x})}{|\vec{x}|}dS_x - \frac{m}{4\pi}\theta(t)\int_{|\vec{x}|\le t}\frac{J_1(m\sqrt{t^2-|\vec{x}|^2})}{\sqrt{t^2-|\vec{x}|^2}}\varphi(\vec{x})d^3x \\
&\overset{(1)}{=} \frac{t}{4\pi}\int_{|\vec{z}|=1}\varphi(t\vec{z})dS_z - \frac{mt^2}{4\pi}\theta(t)\int_{|\vec{z}|\le 1}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{\sqrt{1-|\vec{z}|^2}}\varphi(t\vec{z})d^3z
\end{aligned}$$

Clearly,

$$\lim_{t\to 0^+} g(t) \overset{(2)}{=} 0$$

Now we will consider $g'(t)$

$$\begin{aligned}
\frac{\partial}{\partial t}g(t) &\overset{(3)}{=} \frac{1}{4\pi}\int_{|\vec{z}|=1}\varphi(t\vec{z})dS_z + \frac{t}{4\pi}\int_{|\vec{z}|=1}\frac{\partial}{\partial t}\varphi(t\vec{z})dS_z - \frac{mt}{2\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z + \\
&- \frac{mt^2}{4\pi}\delta(t)\int_{|\vec{z}|\leq 1}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z - \frac{mt^2}{4\pi}\theta(t)\frac{\partial}{\partial t}\int_{|\vec{z}|\leq 1}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z \\
&\overset{(4)}{=} \frac{1}{4\pi}\int_{|\vec{z}|=1}\varphi(t\vec{z})dS_z + \frac{t}{4\pi}\int_{|\vec{z}|=1}\frac{\partial}{\partial t}\varphi(t\vec{z})dS_z - \frac{mt}{2\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z + \\
&- \frac{mt^2}{4\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{\partial}{\partial t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z
\end{aligned}$$

From here, it is easy to see that

$$\lim_{t\to 0^+}\frac{\partial}{\partial t}g(t) \overset{(5)}{=} \delta(t) \text{ in } \mathcal{D}'(\mathbb{R})$$

Now we will analyze $\frac{\partial^2}{\partial t^2}g(t)$.

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}g(t) &\overset{(3,4)}{=} \frac{1}{2\pi}\int_{|\vec{z}|=1}\frac{\partial}{\partial t}\varphi(t\vec{z})dS_z + \frac{t}{4\pi}\frac{\partial}{\partial t}\int_{|\vec{z}|=1}\frac{\partial}{\partial t}\varphi(t\vec{z})dS_z - \frac{m}{2\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z \\
&- \frac{mt}{2\pi}\delta(t)\int_{|\vec{z}|\leq 1}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z - \frac{mt}{\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{\partial}{\partial t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z \\
&- \frac{mt^2}{4\pi}\delta(t)\int_{|\vec{z}|\leq 1}\frac{\partial}{\partial t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z - \frac{mt^2}{4\pi}\theta(t)\frac{\partial}{\partial t}\int_{|\vec{z}|\leq 1}\frac{\partial}{\partial t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z \\
&\overset{(6)}{=} \frac{1}{2\pi}\int_{|\vec{z}|=1}\frac{\partial}{\partial t}\varphi(t\vec{z})dS_z + \frac{t}{4\pi}\int_{|\vec{z}|=1}\frac{\partial^2}{\partial t^2}\varphi(t\vec{z})dS_z - \frac{m}{2\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z \\
&- \frac{mt}{\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{\partial}{\partial t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z - \frac{mt^2}{4\pi}\theta(t)\frac{\partial}{\partial t}\int_{|\vec{z}|\leq 1}\frac{\partial}{\partial t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z \\
&\overset{(7)}{=} \frac{1}{2\pi}\int_{|\vec{z}|=1}\frac{\partial}{\partial t}\varphi(t\vec{z})dS_z + \frac{t}{4\pi}\int_{|\vec{z}|=1}\frac{\partial^2}{\partial t^2}\varphi(t\vec{z})dS_z - \frac{m}{2\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z \\
&- \frac{mt}{\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{\partial}{\partial t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z - \frac{mt^2}{4\pi}\theta(t)\int_{|\vec{z}|\leq 1}\frac{\partial^2}{\partial t^2}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{1-|\vec{z}|^2}\varphi(t\vec{z})d^3z
\end{aligned}$$

So,

$$\lim_{t\to 0^+}\frac{\partial^2}{\partial t^2}g(t) \overset{(8)}{=} 0 \text{ in } \mathcal{D}'(\mathbb{R})$$

(1) $\vec{x} = t\vec{z} \implies dS_x = t^2 dS_z, \quad d^3x = t^3 d^3z$
(2) Both $J_1(x)$ and $\varphi(x)$ are bounded
(3) $\frac{\partial}{\partial t}\varphi(t\vec{z})$ is bounded by definition
(4) $\frac{\partial}{\partial t}\left[\frac{J_1(mt\sqrt{1-|\vec{z}|^2}}{\sqrt{1-|\vec{z}|^2}}\varphi(t\vec{z})\right] = \varphi(t\vec{z})\frac{\partial}{\partial t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{\sqrt{1-|\vec{z}|^2}} + \frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{\sqrt{1-|\vec{z}|^2}}\frac{\partial}{\partial t}\varphi(t\vec{z})$
$= \varphi(t\vec{z})\left[mJ_0(mt\sqrt{1-|\vec{z}|^2}) - \frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{t\sqrt{1-|\vec{z}|^2}}\right] + \frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{\sqrt{1-|\vec{z}|^2}}\frac{\partial}{\partial t}\varphi(t\vec{z}) \leq M_\varphi[m + \frac{m}{2}] + \frac{1}{2}mM_{\varphi'}$
where $M_\varphi = \sup\varphi$ and $M_{\varphi'} = \sup\varphi'$
(5) $\lim_{t\to 0^+}\frac{1}{4\pi}\int_{|z|=1}\varphi(t\vec{z})dS_z = \frac{1}{4\pi}\int_{|\vec{z}|=1}\varphi(0)dS_z = \varphi(0) = (\delta, \varphi)$ by LDCT. The rest trivially go to zero.
(6) $\frac{\partial^2}{\partial t^2}\varphi(t)$ is bounded by definition.

(7) $\frac{\partial^2}{\partial t^2}\left[\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{\sqrt{1-|\vec{z}|^2}}\varphi(t\vec{z})\right] = 2\frac{\partial}{\partial t}\varphi(t\vec{z})\frac{\partial}{\partial t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{\sqrt{1-|\vec{z}|^2}} + \frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{\sqrt{1-|\vec{z}|^2}}\frac{\partial^2}{\partial t^2}\varphi(t\vec{z}) + \varphi(t\vec{z})\frac{\partial^2}{\partial t^2}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{\sqrt{1-|\vec{z}|^2}}$
The first two terms are bounded using similar arguments as shown in $(2, 3, 4, 6)$.
Now, $\varphi(t\vec{z})\frac{\partial^2}{\partial t^2}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{\sqrt{1-|\vec{z}|^2}} = \varphi(t\vec{z})\left[\frac{2m}{t}\frac{J_1(mt\sqrt{1-|\vec{z}|^2})}{mt\sqrt{1-|\vec{z}|^2}} - m^2\sqrt{1-|\vec{z}|^2}J_1(mt\sqrt{1-|\vec{z}|^2}) - \frac{m}{t}J_0(mt\sqrt{1-|\vec{z}|^2})\right]$
$\leq \left[\frac{2m}{t} + m^2\sqrt{1-|\vec{z}|^2}\right]\sup\varphi$
Since $\frac{2m}{t} \leq \frac{2m}{a}$ for $t \in [a, \infty]$, we can bring the partial inside by Fubini's Theorem.
(8) LDCT on certain integrals following from (2-7).

4 Convolution

Observe that the causal Green's function of a (KGF) operator has support in he future light cone Γ^+ where $t \geq |x| \geq 0$. Therefore the generalized cauchy problem has a solution $u(x,t) = (G * h)(x,t)$ for any distribution $h(x,t)$ with support in the positive half-space $t \geq 0$, and the solution is unique.
Let $g(x,t) \in \mathcal{D}'(\mathbb{R}^{3+1})$ satisfying $\operatorname{supp} g \subset \Gamma^+$ and let $u(x) \in \mathcal{D}'(\mathbb{R}^3)$.
Then by our theorem about the convolution of distributions with support in Γ^+ and $t \geq 0$, and from the commutativity and associativity of the direct product,

$$\begin{aligned}
(g(x,t) * (u(x)\cdot\delta(t)), \varphi(x,t)) &= (g(x,t)\cdot(u(y)\cdot\delta(\tau)), \eta(\tau)\eta(t)\eta(t^2-|x|^2)\varphi(x+y,t+\tau)) \\
&= (g(x,t)\cdot u(y)\cdot\delta(\tau), \eta(\tau)\eta(t)\eta(t^2-|x|^2)\varphi(x+y,t+\tau)) \\
&= (g(x,t)\cdot u(y), \eta(t)\eta(t^2-|x|^2)\varphi(x+y,t)) \\
&= (g(x,t)\cdot u(y), \eta(t^2-|x|^2)\varphi(x+y,t))
\end{aligned}$$

Therefore,

$$g(x,t) * (u(x)\cdot\delta(t)) = g(x,t) * u(x)$$

Since the convolution exists, we can also deduce from here that

$$g(x,t) * (u(x)\cdot\delta^{(p)}(t)) = \frac{\partial^p g(x,t)}{\partial t^p} * u(x)$$

Thus, a solution to the (KGF) equation can be written as

$$\begin{aligned}
v(\vec{x},t) &= G(\vec{x},t) * (u_1(\vec{x})\cdot\delta(t)) + G(\vec{x},t) * (u_0(\vec{x})\cdot\delta'(t)) \\
&= G(\vec{x},t) * u_1(\vec{x}) + \frac{\partial}{\partial t}G(\vec{x},t) * u_0(\vec{x})
\end{aligned}$$

We can now verify our initial conditions. By continuity of the convolution of distributions lying in a positive light cone with those having bounded support,

$$\begin{aligned}
\lim_{t\to 0^+} v(\vec{x},t) &= \lim_{t\to 0^+} G(\vec{x},t) * u_1(\vec{x}) + \lim_{t\to 0^+}\frac{\partial}{\partial t}G(\vec{x},t) * u_0(\vec{x}) \\
&= 0 * u_1(\vec{x}) + \delta(t) * u_0(\vec{x}) \\
&= u_0(\vec{x})
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to 0^+}\frac{\partial}{\partial t}v(\vec{x},t) &= \lim_{t\to 0^+}\frac{\partial}{\partial t}G(\vec{x},t) * u_1(x) + \lim_{t\to 0^+}\frac{\partial^2}{\partial t^2}G(\vec{x},t) * u_0(\vec{x}) \\
&= \delta(t) * u_1(\vec{x}) + 0 * u_0(\vec{x}) \\
&= u_1(\vec{x})
\end{aligned}$$

These results and the fact that $v \in C^1(t > 0)$ follow directly from calculations and analysis in section 3. So we have found a valid distributional solution, but does it have bounded support? Our solution having bounded support is both important for physical realism and key for the well-posedness of our problem. In this next section we will show that if u_0 and u_1 have bounded support, then our solution to the generalized Cauchy problem has bounded support in the variable x for any $t > 0$.

5 Hyugen's principle

Observe that for a given $t > 0$, $\operatorname{supp} G(x,t)$ and $\operatorname{supp} \frac{\partial}{\partial t} G(x,t)$ are bounded by a sphere of radius t. So by taking $G(\vec{x},t) * u_1(\vec{x}) + \frac{\partial}{\partial t} G(\vec{x},t) * u_0(\vec{x})$,

$$\operatorname{supp} v(x,t) = \bigcup_{x \in \operatorname{supp} u_1} B_t(x) \cup \bigcup_{x \in \operatorname{supp} u_0} B_t(x)$$

as shown in figure 4. Hence, the support of our solution is bounded in x for any given $t > 0$.

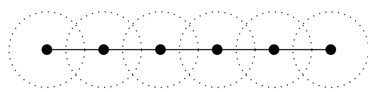

Figure 4: Huygens–Fresnel principle

6 Appendix

$$\frac{1}{2\pi i} \lim_{b \to \infty} \int_{-b}^{b} e^{\pm i m \sqrt{s} \cosh(\xi + \xi_0)} d\xi = \frac{1}{2} \left(J_0(m\sqrt{s}) \pm i N_0(m\sqrt{s}) \right)$$
$$\frac{i}{2\pi} \lim_{b \to \infty} \int_{-b}^{b} e^{\pm i m \sqrt{s} \sinh(\xi + \xi_0)} d\xi = \frac{i}{\pi} K_0(m\sqrt{s})$$

where $s > 0$ in both formulas.